Class 5, given on Jan 13, 2010, for Math 13, Winter 2010

1. Double integrals

Recall that we discussed how a definite integral for a function of two variables $f(x, y)$ over a rectangle R is defined, as the limit of Riemann sums which represented sums of volumes of lots of small rectangular prisms. In particular, the definite integral should represent the (signed) volume under $z = f(x, y)$ over the rectangle R.

Examples.

- Evaluate the double integral
$$\iint\limits_R c\, dA$$
in terms of the area of the rectangle R, where c is some constant. This double integral is equal to the volume of a rectangular prism with base of area $A(R)$ and height c, so this double integral should equal $cA(R)$. This is the two-dimensional analogue of the formula
$$\int_a^b c\, dx = c(b - a).$$
- Using the interpretation of a double integral as volume, calculate the definite integral
$$\iint\limits_R \sqrt{1 - x^2}\, dA$$
where R is the rectangle $[-1, 1] \times [-1, 1]$. We begin by making a sketch of the graph of this function over the rectangle R. Notice that $\sqrt{1 - x^2}$ does not depend on y, so the cross-sections of the graph of this function taken when we fix y will all look identical. In particular, these cross-sections are of the form $z = \sqrt{1 - x^2}$, which is the upper half of a circle of radius 1. So the double integral we are evaluating represents the volume of a region whose cross-sections by planes of the form $y = C$ are all half-discs of radius 1. In other words, our region is half of a cylinder, whose base is a circle of radius 1 and whose length is 2. The volume of such a solid is given by
$$\frac{1}{2}\pi \cdot 2 = \pi.$$

The double integral also has an interpretation as the average value of a function over the rectangle R. In the single-variable case, the expression
$$\frac{1}{b - a}\int_a^b f(x)\, dx$$
represents the average value of $f(x)$ on the interval $[a, b]$; in the two-variable case, the expression
$$\frac{1}{A(R)}\iint\limits_R f(x, y)\, dA$$
represents the average value of $f(x, y)$ on the rectangle R. For example, if R represents the boundaries of a city, and $f(x, y)$ is the amount of rainfall (that is, the number of inches of rain) the city received at the point (x, y), then this average value represents the average amount of rainfall the city received.

2. Iterated integrals and Fubini's Theorem

The previous example (the half-cylinder) provides the clue to answering the question of how to evaluate double integrals. Notice that we solved that problem by recognizing a cross-section of the solid we wanted to calculate the volume of when we cut it with planes of the form $y = C$. Suppose we want to integrate

$$\iint\limits_R f(x, y)\, dA$$

We might approach this problem by taking cross-sections of the solid under $z = f(x, y)$, over the rectangle R, when we fix either x or y. Concretely, suppose $R = [a, b] \times [c, d]$. Suppose we fix x, so we take cross-sections with planes of the form $x = C$. Then the appropriate cross-section then has area given by the formula

$$A(x) = \int_c^d f(x, y)\, dy$$

where in this integral we treat x as a fixed number. For example, if we are interested in evaluating

$$\iint\limits_R x^2 + y^2\, dA$$

where $R = [0, 2] \times [0, 3]$, then

$$A(x) = \int_0^3 x^2 + y^2\, dy = \left(yx^2 + \frac{y^3}{3} \right) \Big|_{y=0}^{y=3} = 3x^2 + 9.$$

In other words, the cross-sectional area of the solid defined by $z = x^2 + y^2$ over the rectangle $[0, 2] \times [0, 3]$ when we fix x is $3x^2 + 9$. We place $y = 0, y = 3$ in the bounds of integration to remind ourselves that we are evaluating the expression $yx^2 + y^3/3$ at the values $y = 0, 3$ instead of $x = 0, 3$. This procedure, where we calculate an integral by fixing one variable, is very strongly reminiscent of partial differentiation; sometimes this is called *partial integration*.

Therefore, it is plausible that the volume of the solid we are interested in can be obtained by integrating the cross-sectional area $A(x)$ with respect to the remaining variable x. That is, we should expect

$$\iint\limits_R f(x, y)\, dA = \int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

From now on, we will drop the parentheses which enclose the inner integral. We call such an expression for a double integral, where we take an integral with respect to one variable first, and then with respect to the remaining variable, an *iterated integral*. In the example we were looking at, then, we should have

$$\iint\limits_R x^2 + y^2\, dA = \int_0^2 3x^2 + 9\, dx = x^3 + 9x \Big|_0^2 = 8 + 18 = 26.$$

There is nothing special about taking cross-sections by fixing x. We could just as well have taken cross-sections by fixing y, and obtained a formula for the cross-sectional area at y:

$$A(y) = \int_a^b f(x, y)\, dx$$

and then integrated this formula over y:

$$\int_c^d A(y)\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

One probably expects that the order of integration should not change the final answer. While this is not always true (see Problem 37 in Chapter 16.2), the following theorem tells us that this is true in virtually every situation we will encounter:

Theorem. (Fubini's Theorem) If $f(x,y)$ is continuous on a rectangle $R = [a,b] \times [c,d]$, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true whenever $f(x,y)$ is bounded on R, continuous on R except possibly at a finite number of smooth curves, and both iterated integrals exist.

You can check that the hypotheses of Fubini's Theorem hold true for basically every example we will look at, and if we encounter any situations where Fubini's Theorem does not hold (which is rather unlikely) this will be explicitly noted.

Notice that once we write an iterated integral down, the ordering of $dx\,dy$ or $dy\,dx$ determines the order in which we should perform the partial integrations. Pay very close attention to the ordering of the differentials, and remember that you begin by integrating with respect to the left-most variable and then work your way to the right. (While we only have two differentials right now, we will be performing triple integrals in the future.)

Examples.

- Let $R = [0,2] \times [0,1]$. Evaluate the double integral

  $$\iint\limits_R xe^y\,dA$$

  using both possible orders of integration. Since Fubini's Theorem is obviously valid, we can write this double integral as either of the iterated integrals

  $$\int_0^2 \int_0^1 xe^y dy\,dx, \quad \int_0^1 \int_0^2 xe^y dx\,dy.$$

  The former integral is equal to

  $$\int_0^2 xe^y \Big|_{y=0}^{y=1} dx = \int_0^2 x(e-1)\,dx = \frac{(e-1)x^2}{2}\Big|_0^2 = 2(e-1).$$

  The latter integral is equal to

  $$\int_0^1 e^y \frac{x^2}{2}\Big|_{x=0}^{x=2} dy = \int_0^1 2e^y\,dy = 2e^y\Big|_0^1 = 2(e-1).$$

  As expected, these two iterated integrals are equal to each other.
- Sometimes it is easier to integrate with respect to one variable first instead of the other variable. For example, let $R = [0,\pi] \times [0,1]$, and evaluate the double integral

  $$\iint\limits_R x\cos(xy)\,dA.$$

  Which variable is it easier to integrate with respect to first? If we want to integrate with respect to x, we will need to perform an integration by parts. However, if we integrate with respect to y, we need only use a quick u-substitution, $u = xy$. Then $du = x\,dy$, and we get

$$\iint_R x\cos(xy)\,dA = \int_0^{\pi}\int_0^1 x\cos(xy)\,dy\,dx = \int_0^{\pi}\left(\sin(xy)\Big|_{y=0}^{y=1}\right)dx = \int_0^{\pi}\sin x\,dx = -\cos x\Big|_0^{\pi} = 2.$$

While in principle it does not matter which variable you integrate with respect to first, in practice it can be computationally easier to integrate with respect to one variable first instead of using the other variable.

- Of course, there is nothing special about the variables x, y. For example, we can evaluate an iterated integral

$$\int_0^2\int_{-1}^1 (u+v)^2\,dv\,du.$$

The first integration gives

$$\int_0^2\left(\frac{(u+v)^3}{3}\Big|_{v=-1}^{v=1}\right)du = \int_0^2 \frac{(u+1)^3}{3} - \frac{(u-1)^3}{3}\,du.$$

This integral is equal to

$$\frac{(u+1)^4}{12} - \frac{(u+1)^4}{12}\Big|_0^2 = \frac{3^4}{12} - \frac{1^4}{12} - \left(\frac{1^4}{12} - \frac{(-1)^4}{12}\right) = \frac{80}{12} = \frac{20}{3}$$

If you want, you can check that this is equal to the answer you would have found had you integrated with respect to u first instead of v.